Stratification of Quantales

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Abstract

We develop a stratified, process-theoretic framework in which quantales encode bulk propagation and stratified restriction induces module and bimodule structures. Exit and entrance paths control directional asymmetries, while Chan–Paton factors arise as boundary-localized module elements. Defects are modeled by bimodules whose failure of strict interchange detects symmetry breaking, with Frobenius reciprocity characterizing transparent interfaces. Symmetry restoration is interpreted as stratified strictification to observable levels, yielding a homotopically compatible description of boundary phenomena independent of metric structure.

Contents

1 Introduction

One of the goals of this paper is to show that stratifications lend themselves quite easily to a lot of physical theories, including causal sets of the type seen in Sorkin's work [8], as well as the factorization algebras of [6]. If one should take away at least one thing from this work, it should be that Treumann-styled exit paths [9] (and the later-coined "entrance paths") can be used to model phenomena such as dimensional criticality and symmetry-breaking across membrane pairs.

Before getting to the heart of its contents, I would like to speculate about many things; namely, that the string coupling constant for open strings ending on (a)symmetric branes should obey the condition:

$$g^n_{\mathrm{Brane}_1} + g^m_{\mathrm{Brane}_2} \approx g_s^{1/2} \tag{1.1}$$

and that under mild hypothesis the spectators of Definition 3.1 could introduce worldsheet anomalies along the string. These are none other than the Chan–Paton defects (CPD) of [4].

This is an unusual suggestion, as traditionally anomalies of this sort are assumed to be confined to membranes themselves. The perspective taken here is that the CPD acts via a 1-form $d\gamma_0$ on the string worldsheet, which introduces "residuals" (due to Frobenius reciprocity violation of the projectors) which prevents higher–homotopy coherence. Should these defects not appear, then $d\gamma_0$ will be a closed form. The worldvolume (up to the critical dimension) swept out by the string will be effectively Lagrangian due to the coupling condition of Eq. (1.1). The asymmetry of brane dimensions should then force a Doppler-like scattering across the string, causing one side to "see" a much greater portion of the coupling, thus leading the other to experience a rarer decay process.

Stratifications

Let X be a topological space.

Definition 1.1. *A* stratification *of X is a continuous map $\mathcal{S} : X \longrightarrow P$ where P is a poset equipped with the Alexandrov topology. For each $p \in P$, the preimage*

$$X_p := \mathcal{S}^{-1}(p)$$

is called the stratum *of $\mathcal{S}$ indexed by p. The sets X_p are collectively called the* strata *of $\mathcal{S}$.*

The following example can be skipped, but shows that stratifications can be physically meaningful:

Example 1.1. *Let $\mathcal{C} = (\mathcal{E}, \prec)$ be a causal set of the type seen in [8] and [2]. Denote the elements of $\mathcal{E}$ by $e_{i\in\mathbb{Z}}$. Then, there is a natural stratification*

$$\mathcal{S}_{\mathrm{Caus}} : \mathbb{L} \longrightarrow \mathcal{C}$$

given by the order $e_{<0} \prec (\ell \equiv e_0) \prec e_{>0}$.

Warning 1.1. *The above is somewhat different in spirit from the causal set program. Rather than starting with a causet (which is equivalent to a T_0 space) and allowing a smooth structure to arise "naturally" as a limit density of events $e_i \in \mathcal{E}$, we instead suppose that there is a topological space $\mathbb{L}$* endowed *with a stratification. Specifically, the strata are the timelike past, present, and future of a double-cone, which we take* locally *to be our spacetime. Thus, $(\mathbb{L}, \ell)$ is a space with basepoint ℓ, which should be interpreted as a gauge parameter encoding an observer's frame-of-reference in a (locally) "covariant" way.*

For simplicity, denote a stratified space simply as X. If X has a basepoint x_0, write (X, x_0).

Proposition 1.1. *Every based space (X, x_0) has a natural stratification induced by x_0.*

Proof. Let $P = [1] = \{0, 1\}$ with $0 < 1$. Define

$$\mathcal{S}_{x_0}(x) := \begin{cases} 0 & \text{if } x = x_0, \\ 1 & \text{if } x \in X \setminus \{x_0\}. \end{cases}$$

This is continuous for the Alexandrov topology on P, and is called the *trivial stratification.* □

Let us now introduce the following definition, due to [9]:

Definition 1.2 (Treumann)**.** *Let $\gamma : [0, 1] \to X$ be a path in a stratified space $(X, \mathcal{S})$. Suppose*

$$\gamma(0) \in X_p \qquad \textit{and} \qquad \gamma(1) \in X_{p'}.$$

If $\dim(X_{p'}) \leq \dim(X_p)$, then γ is called an exit path *of X.*

Notation 1.1. *The category of all exit paths in a stratified X with codimension at most n shall be denoted* $\mathsf{Exit}_{\leq n}(X)$*.*

Warning 1.2. *The notation above differs from Treumann's original usage, where* $\mathsf{EP}_{\leq n}$ *denoted an exit-path n-category. We might expect at least* some *agreement between the two categories, however, by considering our category as a stratified fundamental n-groupoid.*

Remark 1.1. *In particular, the trivial stratification is witnessed by an exit path* $\gamma \in \mathsf{Exit}_{\leq 0}(X)$*.*

Lemma 1.1. *Let X be a manifold with corners. Then there is a natural stratification given by corner depth (equivalently, by face codimension), and every path ending on a deepest corner is an exit path.*

Proof. It suffices to illustrate the canonical model $X = \Delta^n$. The faces of Δ^n are naturally indexed by their codimension. Let $P = [n] = \{0, 1, \ldots, n\}$ with $0 < 1 < \cdots < n$, where k corresponds to codimension k. Define $\mathcal{S}_{\Delta^n} : \Delta^n \to P$ by sending a point to the codimension of the unique open face containing it. Thus:

- $\mathcal{S}_{\Delta^n}(x) = 0$ on the interior,
- $\mathcal{S}_{\Delta^n}(x) = 1$ on the open $(n-1)$-faces,
- $\ldots$,
- $\mathcal{S}_{\Delta^n}(x) = n$ at the vertices (the "corners").

If γ is a path with $\gamma(1)$ a vertex, then $\gamma(1)$ lies in a stratum of dimension 0, while $\gamma(0)$ lies in a stratum of dimension ≥ 0. Hence $\dim(X_{p'}) \leq \dim(X_p)$ holds, so γ is an exit path. □

Remark 1.2. *In particular, the deepest corners (vertices) are the maximal-codimension strata. We will later need this lemma for §4.*

Corollary 1. *Every exit path ending on a deepest corner of a manifold with corners lies in* $\mathsf{Routes}_{\leq 0}$ *(equivalently, it is trivial in the sense of this note).*

Why Exit Paths?

Exit paths by now have a relatively mature theory; however, applications are less developed. Ayala–Francis–Tanaka [1] developed their theory of stratifications in relation to the frameworks of TQFTs, and this included the application of a category $\mathsf{Entr}_{\leq n}$, which for all purposes can be treated as the natural dual of the exit path ∞-category.

Armed with both these ∞-categories, one can define (even trivially) an ambient category $\mathsf{Routes}_{\leq n}$, which consists of both the raising and lowering of stratum dimension. If we set $n = 0$, then we get a category

$$\mathsf{Paths}_{\mathcal{S}} = \mathrm{Maps}(\mathcal{S}|_A, \mathcal{S}|_B),$$

where $\dim(A) = \dim(B)$, and there is a homotopy equivalence $A \simeq B$ given by $A \times I \to B$.

We might like to think of all the paths $\gamma \in \mathsf{Paths}_{\mathcal{S}}$ as vibrational modes of a string. Recall our category

$$\mathbf{Strings} = \mathbf{Path} \times \underline{\mathbf{Terms}}$$

from [3]. Since $\mathsf{Paths}_{\mathcal{S}}$ is a subcategory of **Paths**, strings in the former category naturally form a subset of strings in the latter. In fact, this is the special subcategory of strings whose endpoints lie only on diffeomorphic branes. Due to the homotopy relation $A \simeq B$ (which becomes a diffeomorphism $\mathcal{B}_1 \cong \mathcal{B}_2$), we might imagine that there is at least some symmetry at play. In most cases, any 2-cell in $\mathsf{Routes}_{\leq 1}$ which raises or lowers will break this symmetry by hypothesis. Conceivably, this could be used to model states of decreasing or increasing complexity; for instance, dimensional reduction in data analysis.

Quantales

Finally, let us include a key ingredient which will help us define Q-modules (Def. 3.3) later on.

Definition 1.3. *A quantale is a complete lattice $(Q, \leq)$ equipped with an associative binary operation $\otimes$ that distributes over arbitrary joins in each variable:*

$$a \otimes \Big(\bigvee S\Big) = \bigvee_{s \in S} (a \otimes s), \qquad \Big(\bigvee S\Big) \otimes a = \bigvee_{s \in S} (s \otimes a),$$

and equipped with a two-sided unit $\mathbb{1}$ satisfying $\mathbb{1} \otimes a = a = a \otimes \mathbb{1}$.

Remark 1.3. *In what follows, the order on a quantale should be interpreted as an information or accessibility order on strata or paths, while the tensor $\otimes$ models compositional propagation across stratified regions. Noncommutativity reflects the asymmetry between exit and entrance paths.*

Assumption 1.1. *Throughout, we assume the quantale admits a functional calculus supporting square roots.*

2 Quantizations

Let $\mathcal{C}$ denote a classical probabilistic system whose configuration space is stratified. We do not assume that $\mathcal{C}$ admits a global system of equations of motion; instead, we regard $\mathcal{C}$ as specifying local transition data along strata and their interfaces.

Our aim is not to recover classical dynamics, but to encode the compositional structure of such transitions after quantization.

Quantale-valued observables

Let $(Q, \leq, \otimes)$ be a quantale satisfying the assumptions stated above. We interpret elements of Q as quantized transition observables or amplitudes, ordered by refinement or accessibility.

Given strata X and Y, we associate to each ordered pair of reference points (x_i, y_j) an element

$$\delta(x_i \otimes y_j) \in Q,$$

to be interpreted as the quantale-valued contribution of a transition localized near (x_i, y_j). No symmetry between $x_i \otimes y_j$ and $y_j \otimes x_i$ is assumed.

Channel decomposition

We define two formal channels by

$$\hat{q}_1 := \bigvee_{i,j} \delta(x_i \otimes y_j), \tag{2.1}$$

$$\hat{q}_2 := \bigvee_{i,j} \delta(y_i \otimes x_j), \tag{2.2}$$

where the joins are taken over the relevant index sets determined by the stratification. The use of joins reflects the aggregation of indistinguishable local contributions within a stratum.

These channels represent oppositely ordered modes of propagation across strata, and need not coincide when $\otimes$ is noncommutative.

Stratified projection

Let Δ denote the diagonal subspace of the product stratum, regarded as a distinguished interface. We introduce formal projection operators

$$\psi_-, \psi_+ : Q \to Q$$

with respect to the stratification induced by the diagonal inclusion.

Assumption 2.1. *The maps $\psi_\pm$ are assumed to be idempotent, join-preserving endomorphisms of Q.*

We define the quantization of $\mathcal{C}$ by

$$Q(\mathcal{C}) := \psi_-(\hat{q}_1) \otimes \psi_+(\hat{q}_2). \tag{2.3}$$

Notation 2.1. *The symbol ; will be used (when convenient) to emphasize "directional pairing" in the monoidal category underlying Q. In the present note, this is schematic: composition ; coincides with the monoidal product $\otimes$ on underlying elements of Q.*

This expression should be read as a compositional pairing of past-directed and future-directed contributions across the diagonal stratum.

Aggregated form

Using the distributivity of $\otimes$ over joins, the above expression may be formally rewritten as

$$Q(\mathcal{C}) = \bigvee_{i,j} \Big(\psi_-\big(\delta(x_i \otimes y_j)\big) \otimes \psi_+\big(\delta(y_j \otimes x_i)\big) \Big). \tag{2.4}$$

When the quantale admits a functional calculus, we may further regard this as encoding a stratified "norm-like" aggregation of bidirectional contributions, though no metric interpretation is assumed.

Interpretation

The quantization $Q(\mathcal{C})$ is thus not a numerical observable but an element of a quantale encoding:

- stratified localization via $\psi_\pm$,
- superposition via joins,
- directed composition via $\otimes$.

Noncommutativity of $\otimes$ reflects the asymmetry between exit and entrance processes, while the lattice order records refinement of accessible transitions.

In this sense, quantization replaces classical equations of motion with a purely algebraic object capturing the compositional geometry of stratified propagation.

Remark 2.1. *The construction of $Q(\mathcal{C})$ should be regarded as functorial in the stratified data of $\mathcal{C}$ rather than as a numerical quantization procedure.*

Frobenius Reciprocity

Definition 2.1 (Frobenius reciprocity for quantales)**.** *Let $(Q, \leq, \otimes)$ be a quantale and $\psi : Q \to Q$ a join-preserving endomorphism. We say that ψ satisfies Frobenius reciprocity if, for all $a, b \in Q$,*

$$\psi(a \otimes b) = \psi(a) \otimes b.$$

Remark 2.2. *Failure of Frobenius reciprocity measures the obstruction to commuting restriction with composition. In the present framework, such failure encodes symmetry breaking between bulk propagation and diagonal-localized boundary data.*

3 Brane Dynamics

Let us define a category $\mathbf{Spect}(X^\infty)$, called the *spectator* $(\infty, 1)$*-category* (hereafter: X-spectator). The objects of $\mathbf{Spect}(X^\infty)$ shall be paracompact topological spaces, and morphisms are attaching maps $e_i^\alpha : \partial D^\alpha \to X^{\alpha-1}$ (with higher morphisms given by coherent homotopies between attaching maps, taken rel. a specified subcomplex when working with CW-pairs).

Let us return to our category $\mathsf{Routes}_{\leq n}$. Its morphisms are assembled into a "global" attaching map

$$\mathscr{E} := \bigsqcup_{\alpha=0}^{n} e_i^\alpha, \tag{3.1}$$

where $i \in \mathbb{Z}_{\geq 0}$ is a fixed gauge-like label.

We shall call n the *critical dimension* of $\mathcal{E}$.

Definition 3.1. *A finite subspace $X^k_{\mathrm{Fin}} \subset X^\infty$ shall be called an* observable *of the spectator if $k < n$ and Frobenius reciprocity (Def. 2.1) holds with respect to the induced stratification.*

Definition 3.2. *Objects $\mathcal{P} \in \mathbf{Spect}(X^\infty)$ which are not observable, and for which a Q-module representation exists, shall be called* probes.

Assuming X^∞ has corners, then it admits a stratification in the natural way (Lem. 1.1) by considering it as a Δ-complex.[1] The attaching map $\mathscr{E}$ induces exactly this structure, and the observables are the coherent objects living in the n-skeleton of the simplex Δ^∞.

Definition 3.3. *Let $(Q, \leq, \otimes, \mathbb{1})$ be a quantale. A left Q-module is a complete lattice M equipped with an action $\odot : Q \times M \to M$ such that*

$$(a \otimes b) \odot m = a \odot (b \odot m), \qquad \mathbb{1} \odot m = m,$$

and such that $\odot$ preserves arbitrary joins in each variable.

Example 3.1 (Symmetry breaking via brane dimension)**.** *Let X be a stratified space whose strata X_{2k} represent even-dimensional Dp-branes, ordered by inclusion of dimension. Let Q be a noncommutative quantale whose elements encode effective string tension.*

A string is represented by a path

$$\gamma : [0,1] \to X$$

with endpoints lying on plaquettes $\Sigma_a \subset X_{2k}$ and $\Sigma_b \subset X_{2k'}$. When $k \geq k'$, the path γ is an exit path; otherwise, it is an entrance path.

Define $\delta(\gamma) \in Q$ as the quantale-valued contribution of γ, and suppose the tensor $\otimes$ incorporates curvature-dependent corrections on higher-dimensional branes. Then, in general,

$$\psi_-\big(\delta(\gamma)\big) \otimes \psi_+\big(\delta(\gamma)\big) \;\neq\; \psi_+\big(\delta(\gamma)\big) \otimes \psi_-\big(\delta(\gamma)\big),$$

reflecting a symmetry breaking induced by stratified geometry.

Remark 3.1. *This asymmetry may be interpreted as a Doppler-like effect in which curvature of the brane induces a directional increase in effective string tension. No metric or relativistic structure is assumed; the effect is entirely encoded in the noncommutativity of the quantale tensor.*

Remark 3.2. *Assume the projections $\psi_\pm : Q \to Q$ are idempotent, preserve arbitrary joins, and satisfy the (lax multiplicative) inequality*

$$\psi_\pm(x) \otimes \psi_\pm(y) \leq \psi_\pm(x \otimes y) \qquad (x, y \in Q).$$

Then the images $Q_\pm := \psi_\pm(Q) \subseteq Q$ inherit natural left Q-module structures by restriction along $\psi_\pm$: for $a \in Q$ and $m \in Q_\pm$ set

$$a \odot_\pm m := \psi_\pm(a \otimes m) \in Q_\pm.$$

In particular, stratified restriction produces boundary-localized observables (in $Q_\pm$) rather than bulk processes (in Q).

Chan–Paton Factors and Defects

In string-theoretic models, boundary degrees of freedom are traditionally encoded by Chan–Paton factors attached to string endpoints. In the present framework, such factors arise functorially from the module structures induced by stratified restriction, rather than from auxiliary representation spaces.

[1] See [7] for more information.

Let Q denote the quantale of bulk string processes. The idempotent, join-preserving projections

$$\psi_-, \psi_+ : Q \to Q$$

induce natural module structures on their images

$$M_- := \mathrm{im}(\psi_-), \qquad M_+ := \mathrm{im}(\psi_+),$$

with actions defined by

$$a \odot_- m := \psi_-(a \otimes m), \qquad m \odot_+ a := \psi_+(m \otimes a).$$

Elements of these modules are interpreted as boundary-localized data encoding the response of string endpoints to bulk propagation. In this sense, Chan–Paton factors are identified with elements of Q-modules, and composition with bulk processes is mediated by the quantale action.

When a string encounters a distinguished interface, such as the diagonal stratum Δ, both past- and future-facing restrictions are present. The corresponding interface data is supported on

$$M_\Delta := \mathrm{im}(\psi_-) \cap \mathrm{im}(\psi_+),$$

which carries both a left and a right action by Q. If these actions satisfy the interchange law

$$(a \odot_L m) \odot_R b = a \odot_L (m \odot_R b),$$

then M_Δ forms a strict Q–Q bimodule. Such bimodules are interpreted as transparent defects, across which propagation is symmetric and Frobenius reciprocity holds.

More generally, the interchange law may fail, even when the individual actions are well-defined. In this case, the interface data forms only a lax bimodule, and additional coherence is required to describe the composition of boundary interactions. We interpret this laxity as symmetry breaking localized at the defect, potentially necessitating higher or operadic refinements.

Remark 3.3. *If the failure of the interchange law is not detectable by processes supported on the critical stratified skeleton, replacing interface data by its maximal representative on observable strata restores strict bimodule behavior, yielding effective symmetry restoration at the level of Chan–Paton factors.*

4 Factorization Algebras

Thus far, we have discussed stratifications in a relatively low-stakes way. Now, we will up the ante just a little by connecting the work here with Costello–Gwilliam–(Barwick) [6, 5] factorization algebras.

Let X^∞ be a topological space as in Def. 3.1. For concreteness, let it be a cellular complex with attaching maps

$$e_i^k : \Delta^k \longrightarrow X^\infty$$

for every k less than or equal to the critical dimension. Take the interiors

$$\Delta^k \setminus \overline{\Delta^k}$$

to be the opens of X^∞.

Theorem 4.1. *The global attaching map $\mathscr{E}$ from Eq.* (3.1) *endows X^∞ with the structure of a factorization algebra if and only if there is a Weiss cover* $\mathrm{int}(\Delta^k) = \mathscr{W}(\Delta^k)$ *for every* $0 < \alpha < k$*-cell.*

Proof. To go about proving this theorem, first consider that there is a natural prefactorization structure on X^∞ given by the mapping $\alpha \mapsto (\alpha + 1)$, and the lemma 1.1 tells us that we have a convenient stratification induced by the Δ-structure. By considering attaching maps

$$\mathscr{E}|_\alpha = \bigsqcup_{\alpha}^{\alpha+1} e_i^\alpha$$

we can form cell subcomplexes. Consider each point $x_i \in \Delta^k$ with barycentric co-ordinates as an "observable" of our theory. Then, the product space

$$\mathscr{F}_A(X) = \bigotimes_i^\alpha x_i$$

(which is a representation of the factorization algebra) belongs to a subcomplexes Δ^k obtained by restricting the global attaching map $\mathscr{E}$ in the canonical way.

The existence of a Weiss cover for each subcomplex guaruntees that the gluing condition is satisfied, and the proof is complete. □

References

[1] D. Ayala, J. Francis, and H. L. Tanaka. *Local Structures on Stratified Spaces* (2017). arXiv:1409.0501.

[2] R. J. Buchanan. *Causal Set Approach to AQFT* (2026). doi: 10.20944/preprints202601.1518.v1.

[3] R. J. Buchanan. (p, q)*-String Junctions as Interstitial Fields on a Modal Lightcone* (2025). doi: 10.20944/preprints202512.1940.v1.

[4] R. J. Buchanan. and P. M. D. Emmerson. *Phenomenology of Chan–Paton Defects on Open Strings* (2025). 10.22541/au.175407818.81476862/v1

[5] C. Barwick *Factorization algebras in quite a lot of generality* arXiv:2602.01292

[6] K. Costello and O. Gwilliam *Factorization algebras in quantum field theory* `https://ncatlab.org/nlab/files/CostelloGwilliam-FactorizationAlgebras.pdf`

[7] A. Hatcher. *Algebraic Topology* (2001).

[8] R. D. Sorkin. *Causal Sets: Discrete Gravity* (1 Sep, 2003). Report Number: SU-GP-2003/1-2. arXiv:gr-qc/0309009.

[9] D. Treumann. *Exit Paths and Constructible Stacks* (2007). arXiv:0708.0659v1.